Comparing Dynamic Multisector Models *

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Abstract

We construct a multisector DSGE model with input-output linkages and a single type of physical capital that is costly to reallocate across sectors. Our model with intratemporal costs of allocating capital fits sectoral output data as well as existing models with sector-specific capital, both in terms of volatility and comovement with aggregate output as well as pairwise correlations between sectors. The spectra of sectoral output produced by our model is similar to those from the other models. The importance of sectoral shocks in our model is also similar to those from other models. However, our simplified model is much more amenable to extensions involving occasionally-binding constraints than the competitor models that feature large state spaces.

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1 Introduction

The use of multisector models with input-output linkages is widespread in macroeconomics (Horvath (2000), Foerster et al. (2011), Acemoglu et al. (2012), Atalay (2017), Miranda-Pinto (2018), Miranda-Pinto (2019), Carvalho and Tahbaz-Salehi (2018)). Due to a serious curse of dimensionality problem (for example, the smallest example in Foerster et al. (2011) has 26 sectors, the largest over 100), these papers often make one of two compromises. First, they abstract from capital accumulation; as a result they cannot speak to investment movements, the dominant component of business cycle fluctuations, and a feature that is important for comovement between sectors. Second, they use linearized solutions, which do not account for potentially relevant non-linearities in production (see Baqaee and Farhi (2017) and Miranda-Pinto and Young (2019)).

In this paper, we construct a multisector model with input-output linkages and a single type of capital that is costly to reallocate across sectors within a period. Firms in our model decide how much capital to rent *after* productivity is observed. We compare our model's implied sectoral dynamics against two multisector models with sector-specific capital and N endogenous state variables, Foerster et al. (2011) and Carvalho (2007), in which sectoral capital must be installed *before* the realization of productivity.

Using data on sectoral output and a calibrated set of structural parameters (input-output shares, capital and labor income shares, consumption shares), we back out the implied series for sectoral productivity shocks using each model. We then use the variance-covariance matrix of these shocks (assuming shocks are Gaussian) to simulate artificial time series from each model and ask how well the models do at replicating the sectoral output dynamics in the US. The one-capital model with intratemporal adjustment costs of capital delivers volatilities and cross-correlations of sectoral output growth that are closer to the data than the two alternatives, provided the cost of adjustment is of modest size.

Next, we study the implications of the capital technology for the propagation of sectoral shocks. In one of our best fitting calibration, the fraction of short-run macroeconomic fluctuations accounted for by sectoral shocks is comparable to Foerster et al. (2011).

2 The Models

2.1 Sector-Specific Capital

The canonical general multisector business cycle model follows Horvath (2000) and Foerster et al. (2011), which have both intermediate input and investment goods linkages. There are N sectors of the economy indexed by $j = 1, ..., N$. In each sector there is a continuum of homogeneous firms that engage in perfect competition. The representative firm of sector j produces Y_{jt} units of good j at time t, using capital stock K_{jt}, labor L_{jt}, and materials M_{ijt} from all other sectors, using a constant returns to scale Cobb-Dogulas production function:

$$Y_{jt} = Z_{jt} K_{jt}^{\alpha_j} \Big(\prod_{j=1}^{N} M_{ijt}^{\gamma_{ij}} \Big) L_{jt}^{\eta_j},$$

where α_j is the share of capital in production, γ is the cost share of intermediate inputs from sector i in sector's j total production, and $\eta_j = 1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij}$ is the importance of labor in production. Log sectoral productivity Z_j follows an AR(1) process:

$$\log(Z_{jt}) = \xi \log(Z_{jt-1}) + \epsilon_{jt}. \tag{1}$$

The innovations ϵ_{jt} are normally distributed with zero mean and standard deviation $\sigma_{\epsilon,j}$. The persistence of productivity is measured by ξ. The variance-covariance matrix for sectoral shocks is denoted by $\Sigma_{\epsilon\epsilon}$. Sectors accumulate their own capital stock according to:

$$K_{jt+1} = I_{jt} + (1-\delta) K_{jt}, \tag{2}$$

where δ the depreciation rate of capital. New capital in sector j, I_{jt}, is built using investment goods X_{ijt} from all sectors according to:

$$I_{jt} = \prod_{i=1}^{N} X_{ijt}^{\theta_{ij}}, \tag{3}$$

where the parameter θ_{ij} measures the cost share of investment goods from sector i in the value of total investment in sector j. Foerster et al. (2011) calibrate Θ using the 1997 physical capital flow matrix. As an important special case, Carvalho (2007) assumes that Θ is equal to the identity matrix, meaning there is no trade of investment goods across sectors; each sector uses only own-sector goods to build capital.

The representative household consumes the N goods of the economy and provides labor to all the sectors. The discounted expected utility of the household is:

$$E_0 \sum_{t=0}^{\infty} \beta^t \Big(\sum_{j=1}^{N} \frac{C_{j,t}^{1-\sigma} - 1}{1-\sigma} - L_t \Big), \tag{4}$$

where β is the discount factor and σ the intertemporal elasticity of substitution in consumption. $L_t = \sum_{j=1}^{N} L_{jt}$ is the total labor supply of the representative household. Labor is perfectly mobile across sectors. We assume indivisible labor with constant marginal utility of labor as in Kim and Kim (2006) and Foerster et al. (2011). The resource constraint of sector j, for $j = 1, .., N$, is

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} M_{jit} + \sum_{i=1}^{N} X_{jit}. \tag{5}$$

This model has as many endogenous state variables as sectors in the economy. Therefore, if we try to study an economy with more than a few sectors, the curse of dimensionality will arise.

2.2 Semi-Specific Capital

We propose an alternative model with a single endogenous state variable. There is one type of capital that can be imperfectly relocated across sectors. The law of motion of capital is:

$$K_{t+1}^{A} = I_t + (1-\delta)K_t^{A}, \tag{6}$$

where K_t^A is the aggregate capital in the economy at time t.The production of new capital uses the constant returns to scale technology:

$$I_t = \prod_{j=1}^{N} X_{jt}^{\theta_j}, \tag{7}$$

where θ_j is the share of investment goods from sector j in the household's production of new aggregate capital.[1] The resource constraints for sectors j are:

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} M_{ji} + X_{jt}. \tag{8}$$

Finally, the equation describing the intratemporal allocation of capital is:

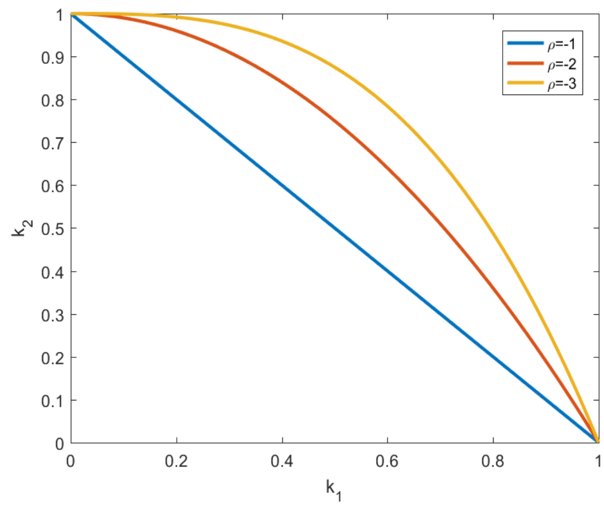
$$K_t^A = \left(\sum_{j=1}^{N} K_{jt}^{-\rho_k}\right)^{-\frac{1}{\rho_k}}, \tag{9}$$

where K_{jt} represents sector's j demand for capital services in period t. Similar to Huffman and Wynne (1999), equation (9) can be interpreted as a reverse CES technology of allocating different mixes of capital stock across sectors. If $\rho_k = -1$, capital is perfectly substitutable across different sectors, so if a positive productivity shock hits the one capital can freely reallocate from less productive industries to that one. If $\rho_k < -1$, capital in different sectors are not perfect substitutes; concentrating capital in one sector generates decreasing marginal improvements in the capital input for that sector. To illustrate this trade off, Figure 1 displays the isoquants for a two sector example of equation (9). We show the isoquants for three different values of ρ_k for the equation $(K_1^{-\rho_k} + K_2^{-\rho_k})^{-1/\rho_k} = 1$.

To understand the quantitative effects suppose first that $\rho_k = -1$. Start with $K_1 = 0.01$ and $K_2 = 0.99$, where the aggregate capital input of the economy K^A is one. Suppose there is a positive productivity shock in sector 1 and the new equilibrium is such that $K_1 = 0.1$ and $K_2 = 0.9$. As sectoral capital freely moves without any cost, aggregate capital input is still one. However, take

[1] Note that in our model, θ is a vector of dimension N containing the shares of investment goods from sector j in the household's production of new capital. On the other hand, in Foerster et al. (2011) Θ is the capital flow matrix (dimension N by N) that contains the investment goods linkages between sectors.

Figure 1
Intratemporal adjustment cost: a two sector example.[a]



[a]Note: Isoquant for $K^A = 1$ with different values of the intratemporal adjustment cost ρ_k.

the same example but with $\rho_k = -2$. In this case, capital reallocation generates a reduction of aggregate capital input from $K^A = 0.9901$ to $K^A = 0.905$, roughly a 10 percent decline in capital input; with an aggregate elasticity of output with respect to capital of roughly 1/3, this decline would translate into a 3.3 percent reduction in aggregate output.

2.3 The Dynamics of Sectoral Output

We make use of the second fundamental welfare theorem and solve the planning problem, since our model economies are Pareto efficient (see Online Appendix for details on the solution method).[2] In the two models in section 2.1 and 2.2, using a local linear approximation we can write the vector of sectoral output growth as:

$$Y_{t+1} = \varrho Y_t + \Xi Z_t + B_z Z_{t+1}, \tag{10}$$

where ϱ, Ξ, and B_z are matrices that depend on the models' parameters, policy rules, and law of motion of states. Assuming the logarithm of sectoral shocks in (1) follows a random walk process ($\xi = 1$), sectoral output growth follows a VARMA(1,1) process:

$$\Delta Y_{t+1} = \varrho \Delta Y_t + \Xi \epsilon_t + B_z \epsilon_{t+1}. \tag{11}$$

Using the models' policy rules we simulate artificial series from the models. To this end, we need to calibrate the model and then characterize the variance-covariance structure for sectoral

[2]Online Appendix: https://drive.google.com/file/d/1mDzI7XqWgzsAfp0FoFFjxiNmJ0c8ss1R/view.

productivity shocks $\{\epsilon_{jt}\}_{t=1}^{T}$. We follow Foerster et al. (2011) and back out the sequences sectoral of productivity shocks according to:

$$\epsilon_{t+1} = B_z^{-1}\Delta Y_{t+1} - B_z^{-1}\varrho\Delta Y_t - B_z^{-1}\Xi\epsilon_t. \tag{12}$$

Provided the system is invertible, we can calculate the implied variance-covariance matrix of shocks $\Sigma_{\epsilon\epsilon}$. If $\Sigma_{\epsilon\epsilon}$ is diagonal, then all shocks in the model economy are purely idiosyncratic; with non-zero off-diagonal elements, there exist "common" shocks that affect productivity in more than one sector.

3 Quantitative Assessment

We calibrate each model's intermediate input shares (Γ), labor shares (η), and capital shares (α) using the 1997 industrial sector input-output table at the two digits industry classification level (26 sectors). The capital flow matrix, Θ, is obtained from the BEA in 1997 and is adjusted as in Foerster et al. (2011) to account for maintenance costs.[3] The vector of investment goods' shares for our model (θ) is calculated as follows. We aggregate up all the sectors as one large sector using investment goods. Then, we re-calculate the fraction of investment goods that this large sector uses from all other sectors. The parameter ρ_k in the single capital model is calibrated to minimize a GMM criteria between the model and the data VAR estimates for sectoral output growth. The rest of parameters follow Foerster et al. (2011), for quarterly data, such as $\delta = 0.025$ and $\beta = 0.99$.

We now proceed to compare the models' implied VAR moments with the ones observed in the data. For each model, we run $S = 500$ simulations of series of size $T = 500$ and estimate a VAR(1) for sectoral output growth. Then, we average the S estimates for the VAR(1) coefficients and variance covariance matrix. We define a GMM criteria as the metric to compare the models' performance in fitting the observed sectoral comovements. We define the model i GMM criterion

$$GMM_i = (\tilde{\varphi}(m_i) - \hat{\varphi})^T W (\tilde{\varphi}(m_i) - \hat{\varphi}), \tag{13}$$

where $\tilde{\varphi}(m_i)$ is the vector of VAR(1) estimates implied from model i, and $\hat{\varphi}$ is the vector of VAR estimates from the observed demeaned sectoral output growth. To capture how well the models fit the implied impulse responses, besides including the VAR correlation coefficients, we include in φ the Cholesky variance covariance estimates. We use two different weighting matrix W: The identity matrix and the inverse of the asymptotic covariance matrix of the VAR(1) estimates using the data.

Table 1 shows that our model with perfectly mobile capital performs the worst, while the model in Foerster et al. (2011) – with sectoral specific capital and investment goods linkages – does not

[3] Foerster et al. (2011) follow McGrattan and Schmitz (1999) and add maintenance costs of capital to the diagonal elements of Θ.

Table 1
GMM criteria value (GMM_i)[a].

Models	W=Identity	W=Asymptotic
Foerster et al (2011)	28.04	2.36E+05
Carvalho (2007)	28.21	2.34E+05
One K model ($\rho_k = -1$)	28.36	2.41E+05
One K model ($\rho_k = -1.2$)	27.66	2.29E+05
One K model ($\rho_k = -1.5$)	27.86	2.32E+05
One K model ($\rho_k = -2.1$)	28.11	2.39E+05

[a]Note: The models VAR coefficients are the average of $S = 500$ simulations of size $T = 500$. The data and model series are demeaned growth rates. Source: Board of Governors industrial production series.

clearly outperform the model in Carvalho (2007) that abstracts from investment goods linkages. Our model with an intratemporal adjustment cost of $\rho_k = -1.2$ is the one with the lowest GMM criteria, regardless the weighting matrix.

In our online Appendix we show that the results in Table 1 also hold in annual data from KLEMS. We also show that our model does at least as good a job matching the entire distribution of pairwise correlations between sectors, not merely the correlations with aggregate output.

These results highlight the importance of including imperfectly-mobile physical capital in our model. In addition, the data suggests that a model that is simpler – compared to Foerster et al. (2011) our model has $N(N-1)-1$ fewer parameters – is able to even outperform models with sector specific capital and investment goods linkages.[4]

4 Spectral Densities

While the models produce similar second moments, that does not necessarily mean they produce these variances at the same frequencies. The spectral density decomposes the volatility of a stationary process into contributions at different frequencies. If sectoral productivity follows a random walk process, the models have a closed form solution for the spectral density function of output growth:

$$S_{\Delta Y}(\omega) = \frac{[I + \Xi e^{-iw}]B_z' \Sigma_{\epsilon\epsilon} B_z [I + \Xi e^{iw}]'}{2\pi [I - \varrho e^{-iw}][I - \varrho e^{iw}]'}. \tag{14}$$

The matrix $\Sigma_{\epsilon\epsilon}$ is the variance-covariance matrix of the implied sectoral productivity shocks in equation (12). The rest of matrices are a function of the models' parameters and policy rules. For a given sequence ω, $S_{\Delta Y}(\omega)$ is an square matrix of dimension N^2 whose diagonal element

[4]Similar results hold if we restrict our sample to the Great Moderation period (1985-2007).

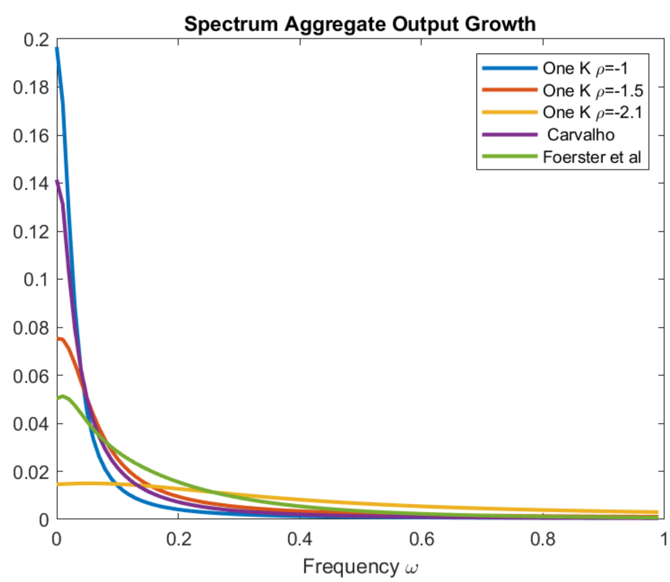
(j,j) contains the contribution of frequency ω to the total variance of sector's j output growth. Hence, we can measure what fraction of the models' variance of output growth lies in the business cycles frequency by calculating the area below the spectrum at the frequencies $\omega \in [0.2, 1]$, which in years corresponds to $[1.5, 8]$. To compare the spectral density of aggregate output growth, we follow Dupor (1999):

$$S_{\Delta \bar{Y}}(\omega) = H^T S_{\Delta Y}(\omega) H, \tag{15}$$

where $\bar{Y}$ is aggregate output and H is a $N \times 1$ matrix of sectoral shares. We assume that sectoral shocks are the only source of aggregate fluctuations. Hence, for each model we use the diagonal of the variance covariance implied by equation (12).

The results in Figure 2 show that the model without intratemporal adjustment costs has a spectral density of aggregate output growth that is very similar to the one in Carvalho (2007), except with a larger variance at low frequencies. Both models display a large fraction of the variance of output growth concentrated in frequencies $\omega \in [0.05, 0.2]$, meaning medium-term cycles of longer than seven years. The model in Foerster et al. (2011) delivers the largest variance of aggregate output at the business cycle frequencies, $\omega \in [0.2, 1]$.

Figure 2



We also show the spectral density for different degrees of intratemporal adjustment costs ($\rho_k < -1$). Our model with $\rho_k = -1.5$—in Table 1, this model still outperforms the models with sector-specific capital—delivers a spectrum between Carvalho (2007) and Foerster et al. (2011).

5 Importance of Sectoral Shocks

Finally, we study the role of sectoral shocks in our model compared to Foerster et al. (2011) and Carvalho (2007). We follow Atalay (2017) and study the average pairwise correlation of the models' implied sectoral shocks from Equation (12):

$$\bar{\rho_\epsilon} = \sum_{i=1}^{N}\sum_{j=1}^{N} corr(\epsilon_i, \epsilon_j).$$

A low average pairwise correlation indicates that idiosyncratic shocks play an important role in accounting for observed business cycle dynamics. Table 2 reports $\bar{\rho_\epsilon}$ for the different models. As expected, the model with investment goods linkages in Foerster et al. (2011) predicts a more dominant role for sectoral shocks ($\bar{\rho_\epsilon} = 0.1205$) compared to the model in Carvalho (2007) ($\bar{\rho_\epsilon} = 0.1698$).

Table 2
Average pairwise correlation of shocks ($\bar{\rho_\epsilon}$)

Models	$\bar{\rho_\epsilon}$
Foerster et al (2011)	0.1205
Carvalho (2007)	0.1698
One K model ($\rho_k = -1$)	0.0903
One K model ($\rho_k = -1.2$)	0.1083
One K model ($\rho_k = -1.5$)	0.1230
One K model ($\rho_k = -2.1$)	0.1277

Our model with an adjustment cost parameter $\rho_k = -1.5$ ($\bar{\rho_\epsilon} = 0.1230$) is quite similar to the model in Foerster et al. (2011). The importance of sectoral shocks in our model decreases when the adjustment cost increases, as capital becomes more difficult to reallocate.

6 Conclusion

We have proposed a reduced state space multisector DSGE model with single capital that is costly to reallocate across sectors. We find that the one-capital model with intratemporal adjustment costs matches the volatility and comovement of sectoral output growth in the US at least as well as prominent alternatives. The benefit of our reduced state space model is that it can more easily be solved globally (it has $N+1$ states instead of $2N$, and all but one of them are exogenous). Thus, we expect that our results will facilitate the study of potentially-important non-linearities, such as occasionally-binding sectoral constraints or non-unitary elasticities in production, in an environment with physical capital accumulation and sectoral linkages.[5]

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[5] Brumm and Scheidegger (2017) is a recent paper that attacks high-dimensional, highly-nonlinear models using adaptive grid methods. The method used there is fairly complicated and requires computational resources inaccessible to many researchers.

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